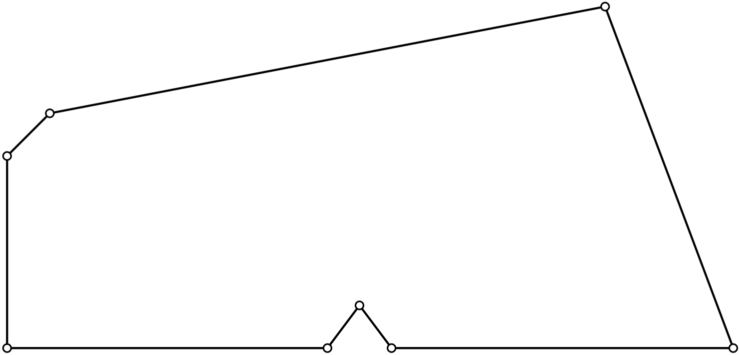
STRAIGHT SKELETONS OF MONOTONE POLYGONS

Therese Biedl   Martin Held   Stefan Huber
Dominik Kaaser   Peter Palfrader

EuroCG 2014, Ein Gedi, Israel

Straight Skeletons

- Aichholzer, Aurenhammer, Alberts, Gärtner 1995[2].
- *Wavefront propagation*: shrinking, mitered offset curves of polygon $\mathcal{P}$.
- Traces of wavefront vertices are the edges of the straight skeleton $\mathcal{S}(\mathcal{P})$.



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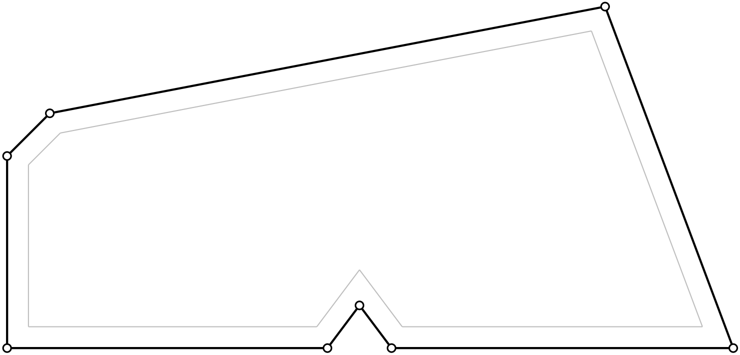

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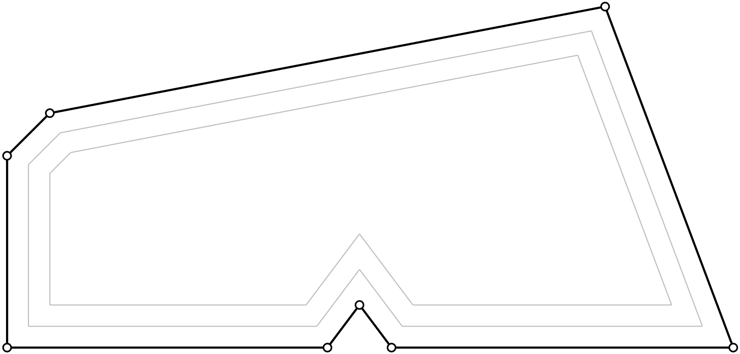

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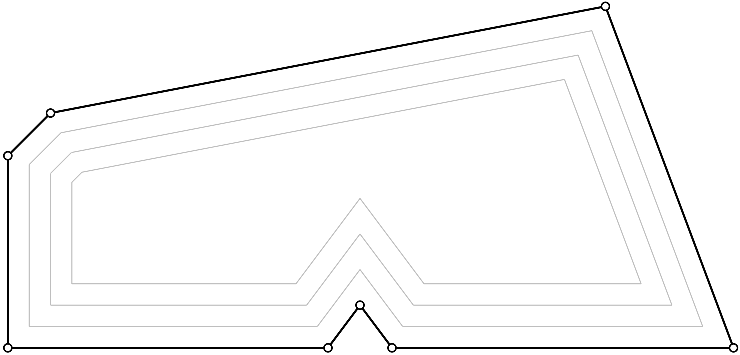

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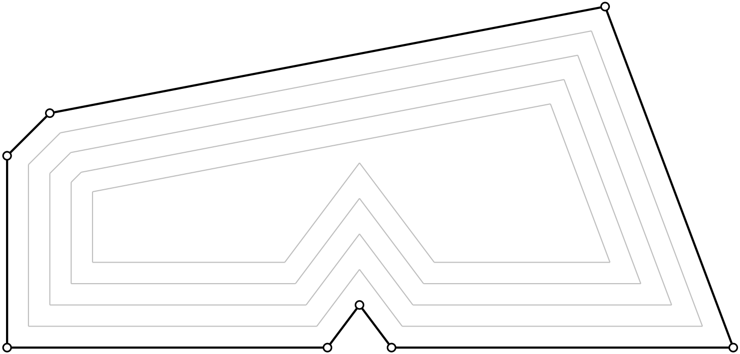

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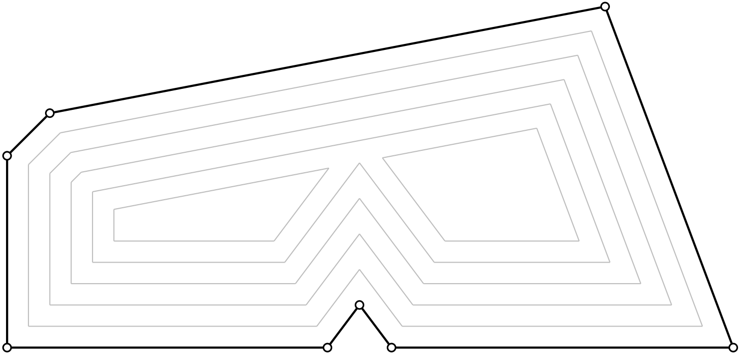

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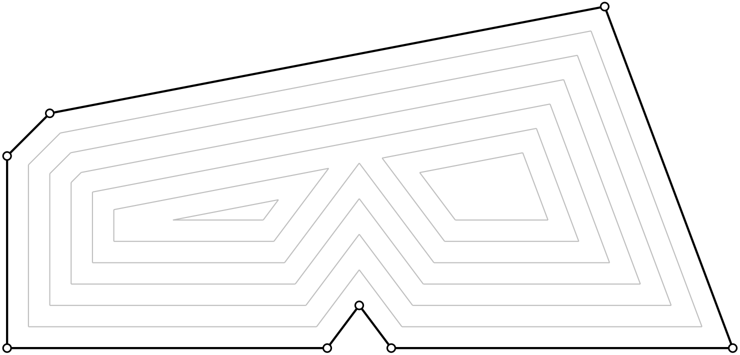

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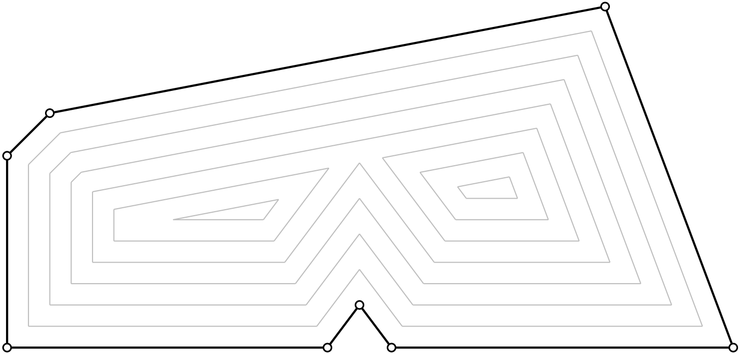

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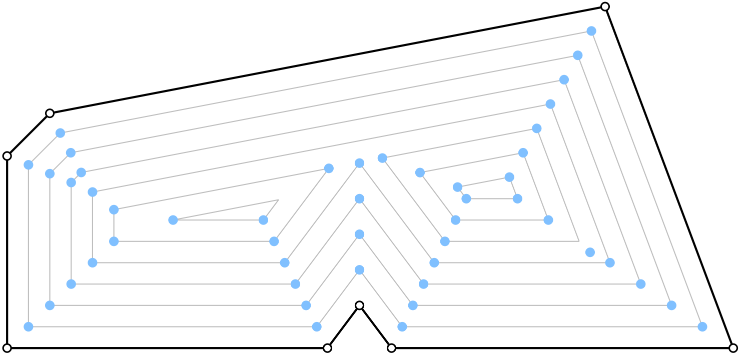

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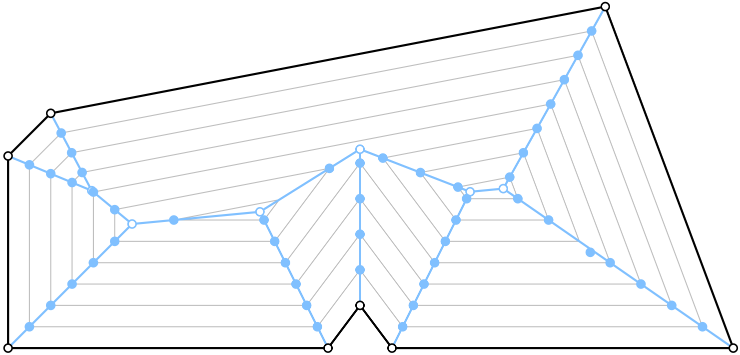

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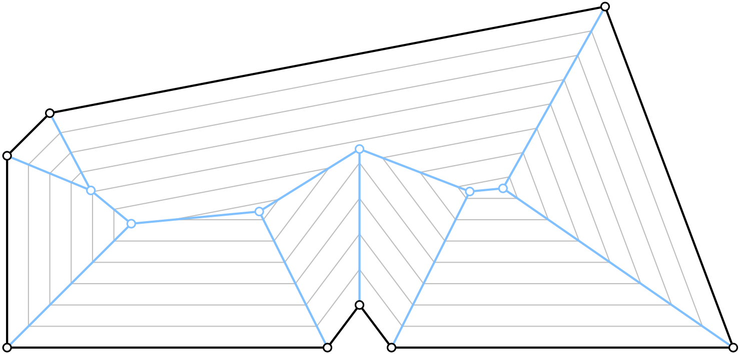
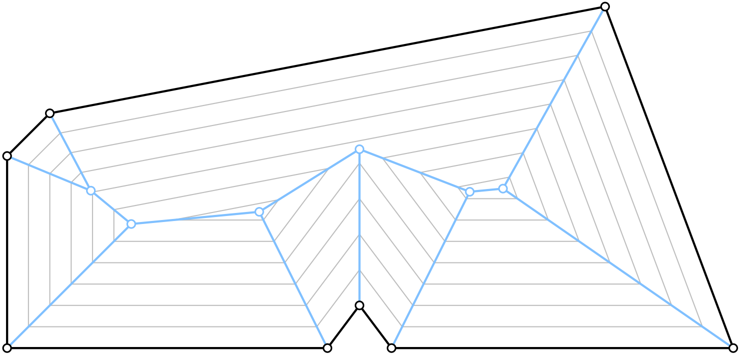


Topology Changes – Edge and Split Events

- Wavefront topology changes over time.
- *edge event*: an edge of the wavefront vanishes.
- *split event*: wavefront splits into two parts.
- In $\mathcal{S}(\mathcal{P})$, events (topology changes) are witnessed by *nodes*.



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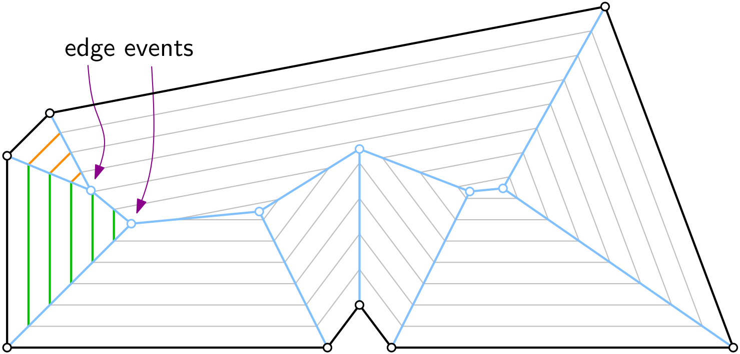

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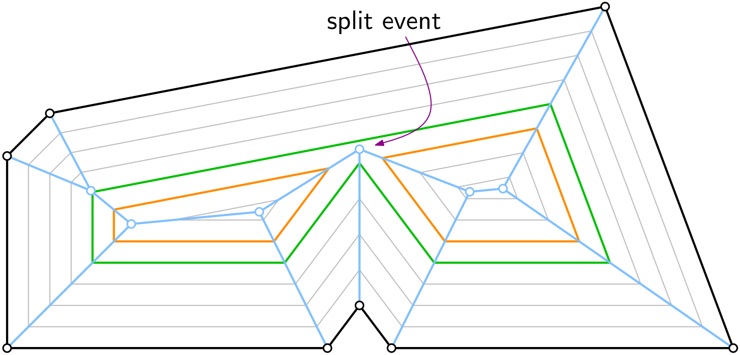

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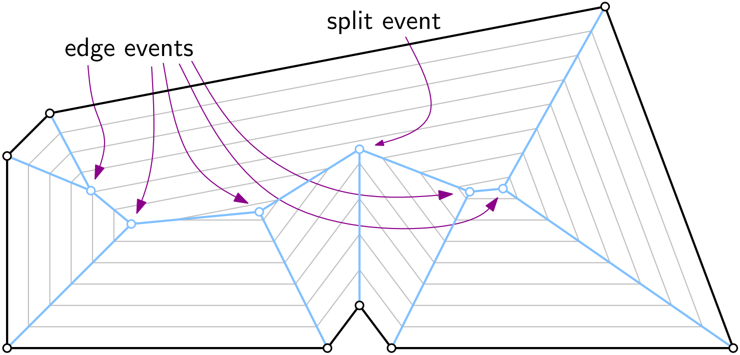

CONSTRUCTING THE STRAIGHT SKELETON

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- Popular approach: Simulate the wavefront propagation.
- Main Problem: Identify next event.
- Edge events are cheap. Split events are expensive.

Can we do better for specific input classes?

YES, FOR (STRICTLY) MONOTONE POLYGONS.

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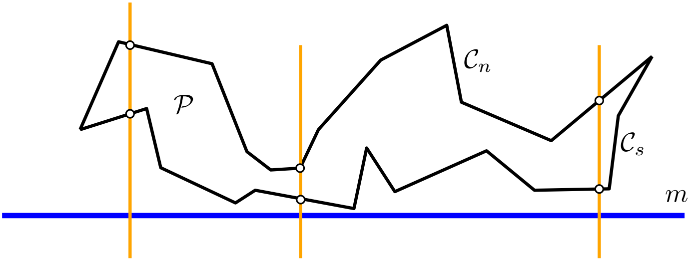
Yes, for (strictly) monotone polygons.

MONOTONE POLYGONS

- Strictly monotone chain $\mathcal{C}$ (monotone w.r.t. to a line *m*):
  Polygonal chain that intersects normals of *m* in at most one point.

- Strictly monotone polygon $\mathcal{P}$ (monotone w.r.t. to a line *m*):
  Simple polygon that can be split into two strictly monotone chains.



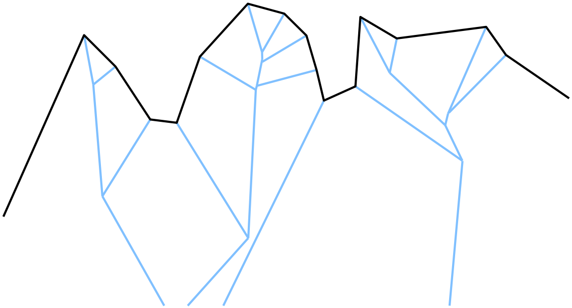
Monotone Polygons - Prior Work

Das et al. claim $\mathcal{O}(n \log n)$ time algorithm[4]:

- Requires general position.
- Correctness?

LEMMATA I

- The wavefront propagation of strictly monotone polygonal chain $\mathcal{C}$ changes only when edges collapse.
- In particular, the wavefront never splits into parts.



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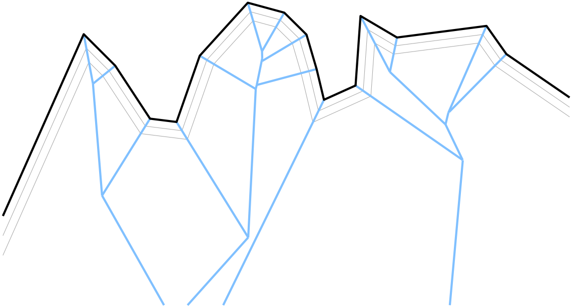

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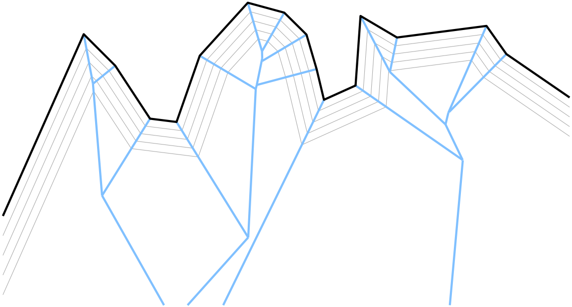

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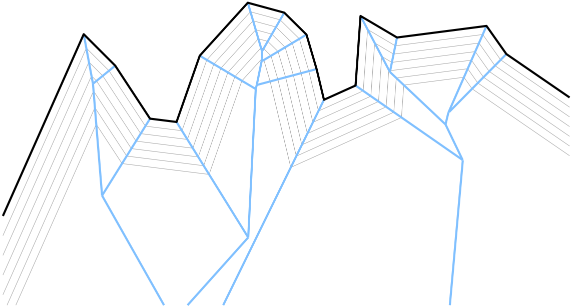

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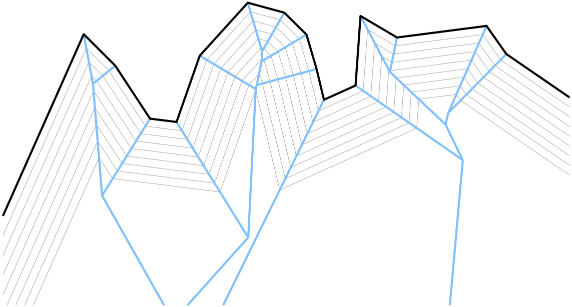

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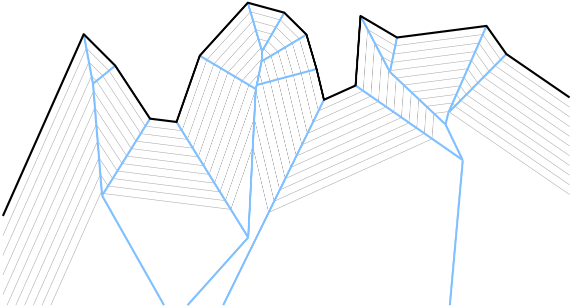

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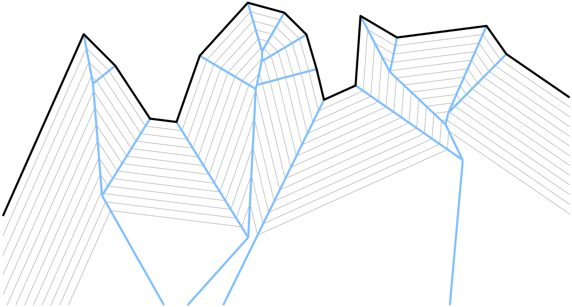

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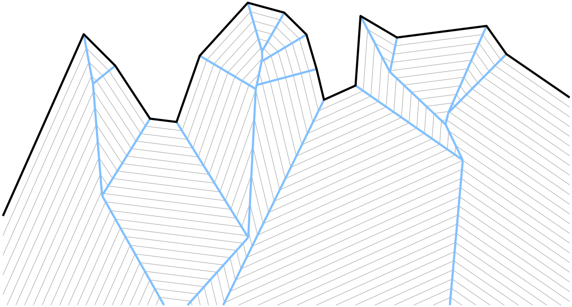

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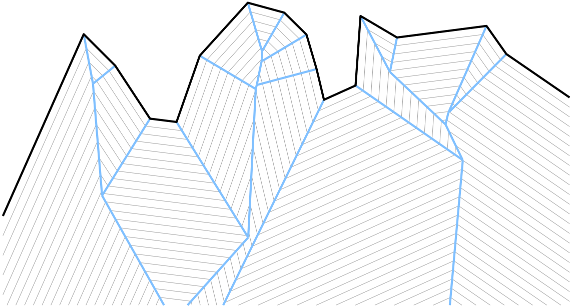

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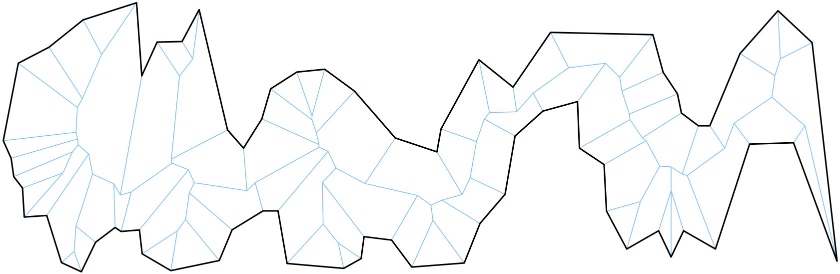

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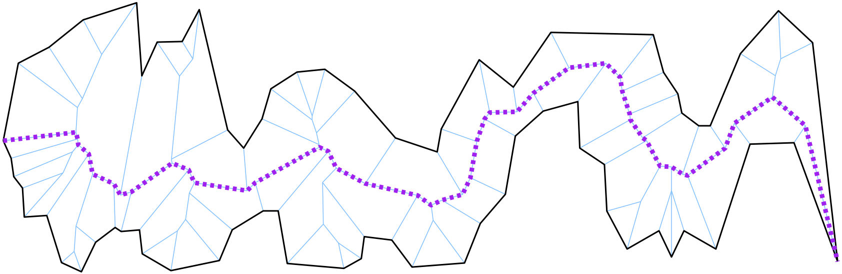

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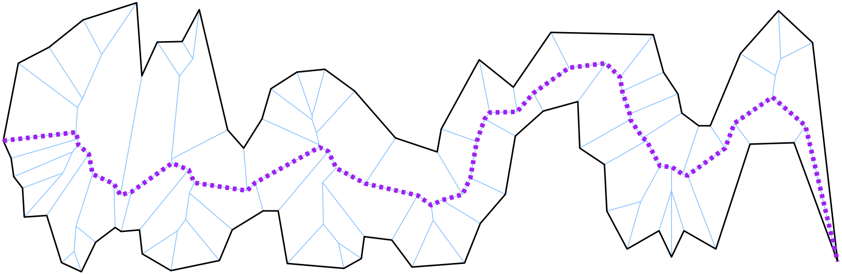

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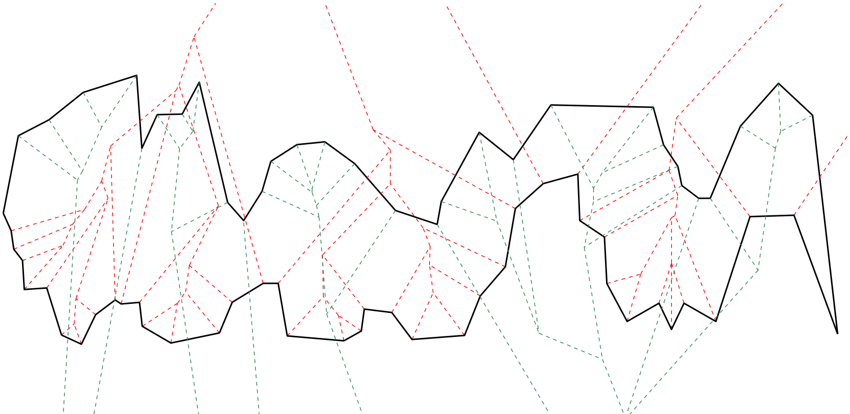

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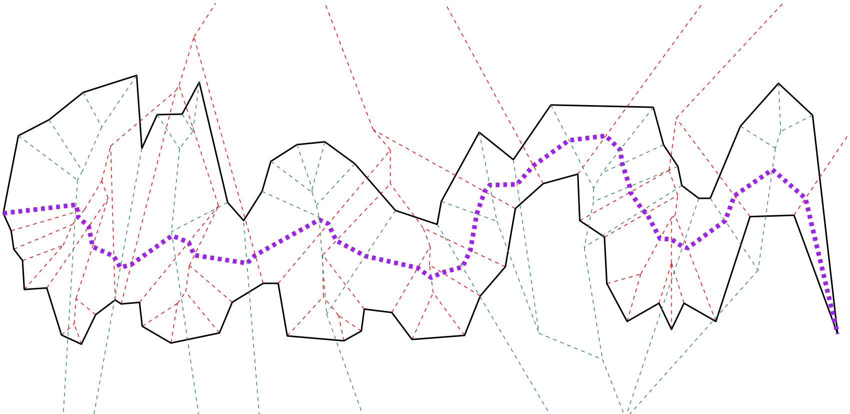

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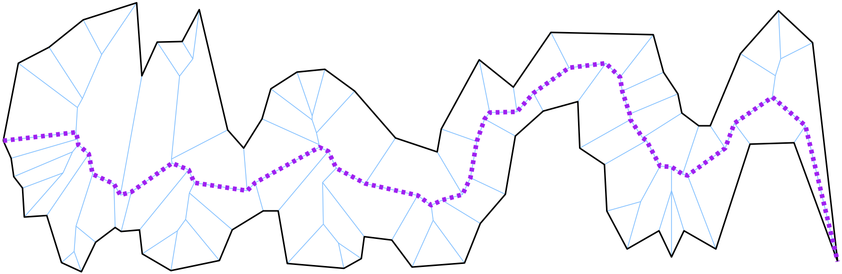

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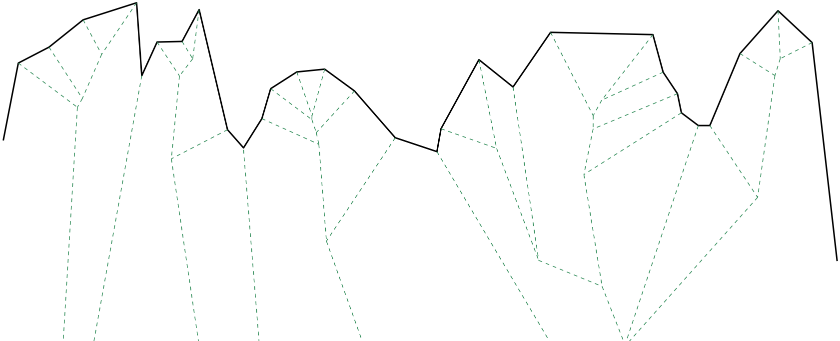

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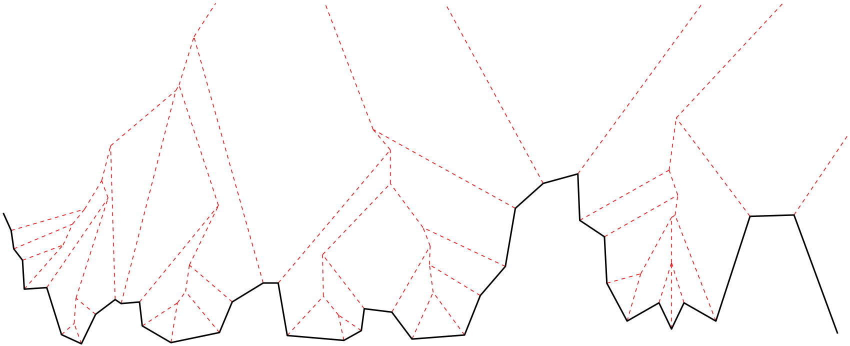

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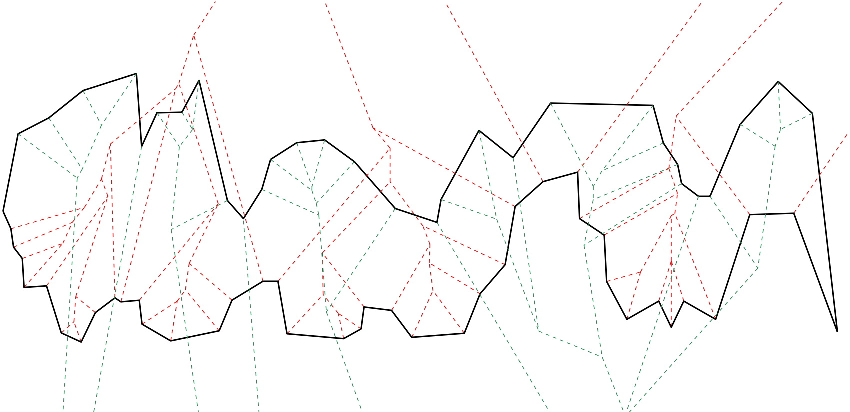

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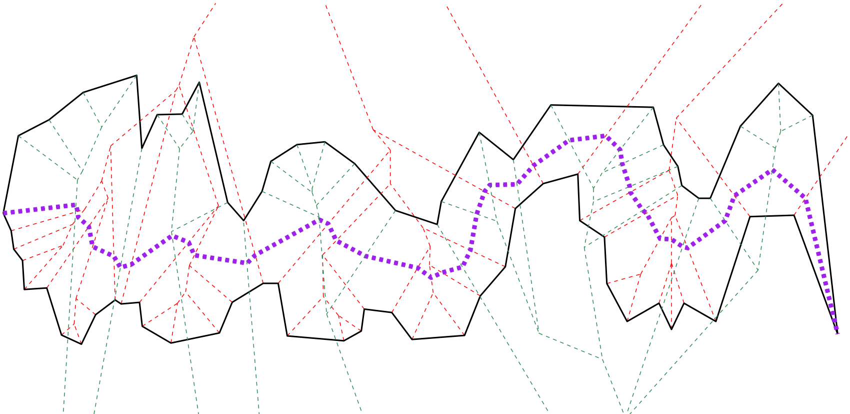

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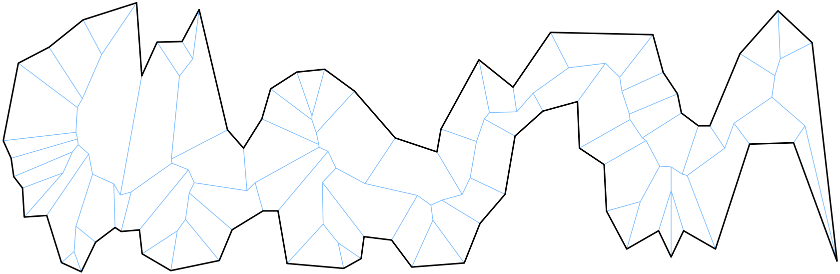

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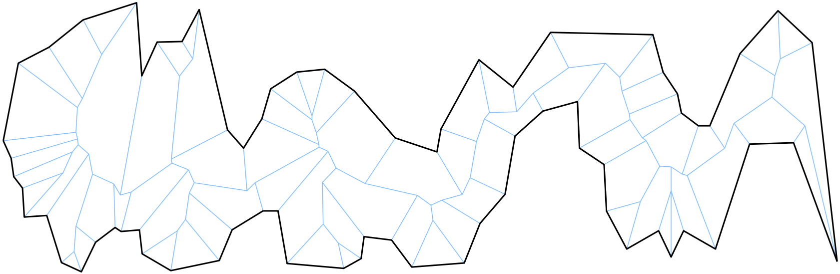
THEOREM:

We can construct $\mathcal{S}(\mathcal{P})$ of strictly monotone polygons $\mathcal{P}$ in time $\mathcal{O}(n \log n)$.

- This also works for the positively weighted straight skeleton $\mathcal{S}_\sigma(\mathcal{P})$.



Algorithm

- Independently construct the north and south skeletons.
- Construct $\mathcal{M}$ and clip the north and south skeletons.
- Assemble $\mathcal{S}(\mathcal{P})$ out of these parts.

Theorem:

We can construct $\mathcal{S}(\mathcal{P})$ of strictly monotone polygons $\mathcal{P}$ in time $\mathcal{O}(n \log n)$.

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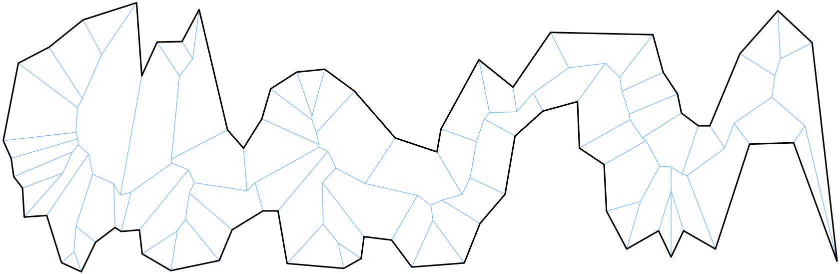

Straight Skeletons of Monotone Polygons

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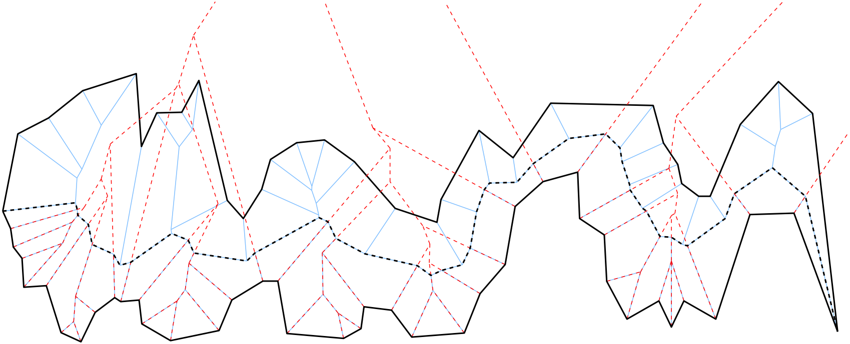

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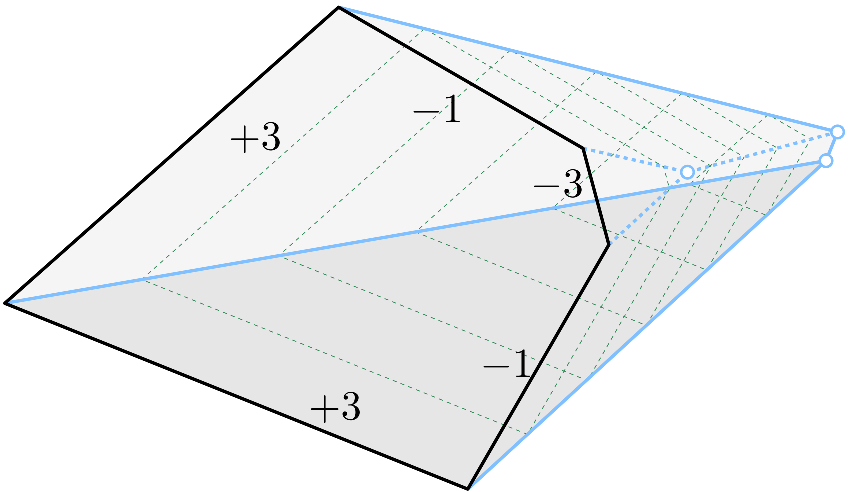
Negative weights



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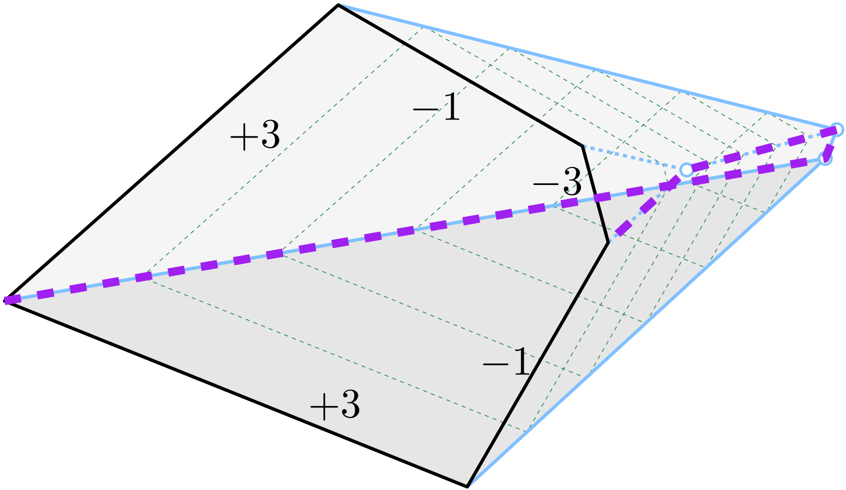
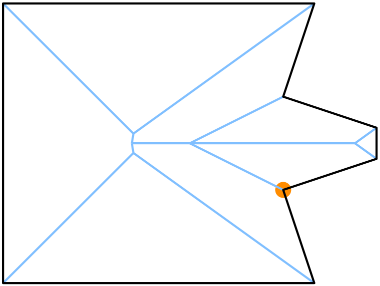


Das et al.[4]: claim $\mathcal{O}(n \log n)$ time algorithm:

- Cannot handle vertex events (perturbation cannot work).
- Wrongly assumes that split event nodes are located at an offset that is the distance to their closest supporting line (similar to Felkel[6]).



MONOTONE POLYGONS - DAS ET AL.

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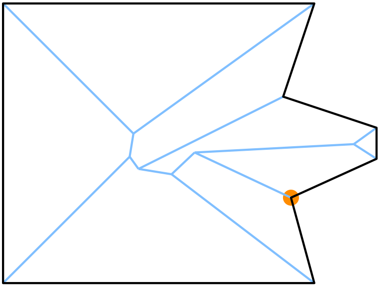

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