Stable Roommates for Weighted Straight Skeletons

Therese Biedl[1] Stefan Huber[2] Peter Palfrader[3]

[1]David R. Cheriton School of Computer Science
University of Waterloo, Canada

[2]Institute of Science and Technology Austria

[3]FB Computerwissenschaften
Universität Salzburg, Austria

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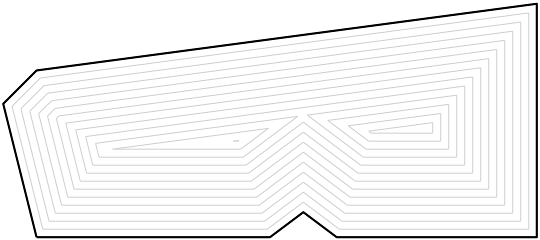
Straight skeletons — a brief introduction



- Introduced by [Aichholzer et al., 1995].

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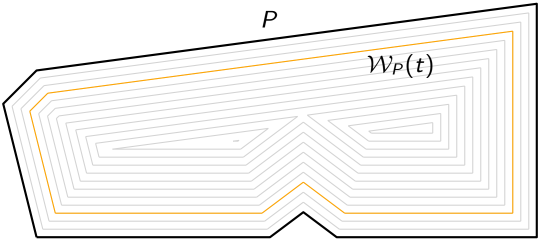

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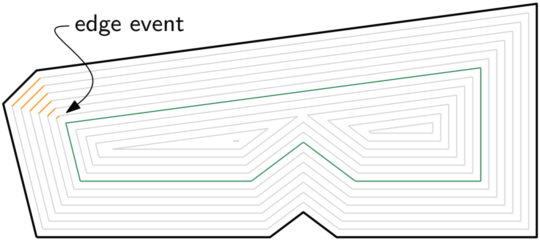

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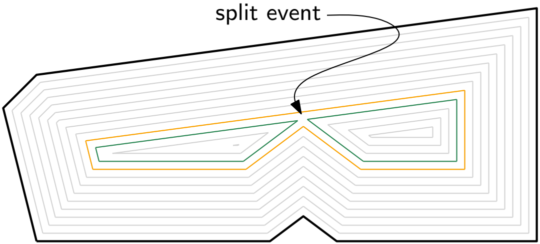

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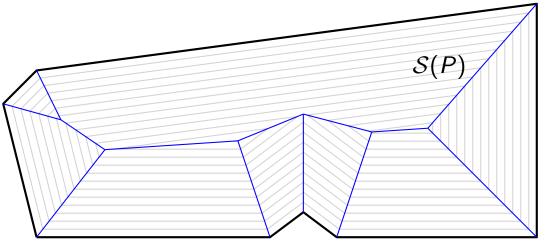

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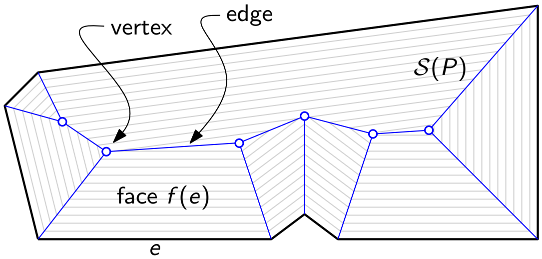

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Straight skeletons — with weights

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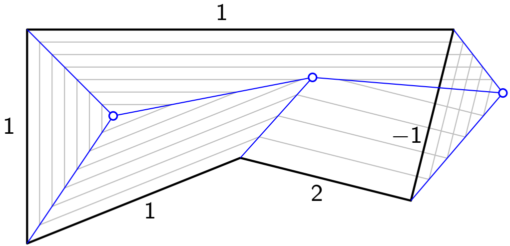
- To every edge e of P a weight $\sigma(e)$ is assigned, its speed.



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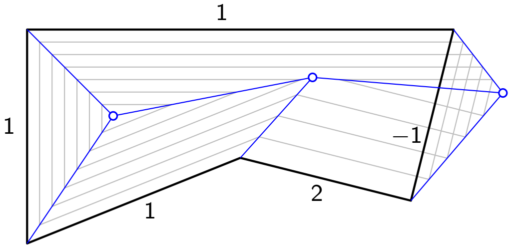

Weighted straight skeletons are "quite established":

- Algorithms were published.
- Implementations are available.
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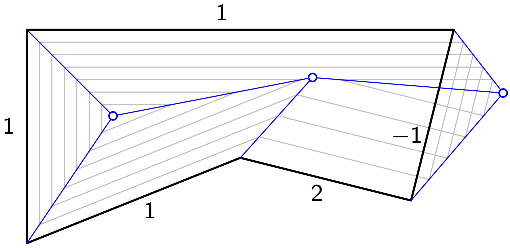

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Still no rigorous definition is known so far!

Prior work

Only since recently we know: weighted straight skeletons can behave very differently.

Property	Simple polygon			Polygon with holes		
	$\sigma \equiv 1$	σ pos.	σ arb.	$\sigma \equiv 1$	σ pos.	σ arb.
$\mathcal{S}(P)$ is connected	✓	✓	✓	✓	✓	×
$\mathcal{S}(P)$ has no crossing	✓	✓	×	✓	✓	×
$f(e)$ is monotone w.r.t. e	✓	×	×	✓	×	×
bd $f(e)$ is a simple polygon	✓	✓	×	✓	×	×
$\mathcal{T}(P)$ is z-monotone	✓	✓	×	✓	✓	×
$\mathcal{S}(P)$ has $n(\mathcal{S}(P)) - 1 + h$ edges	✓	✓	×	✓	✓	×
$\mathcal{S}(P)$ is a tree	✓	✓	×			

Table : [Biedl et al., 2013]

Prior work — ambiguity of edge events

Ambiguity for parallel edges of different weights become adjacent.

Figure : Resolution methods proposed in [Biedl et al., 2013].

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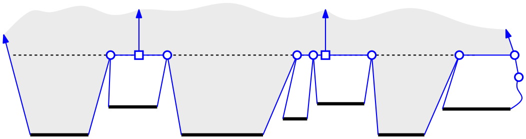

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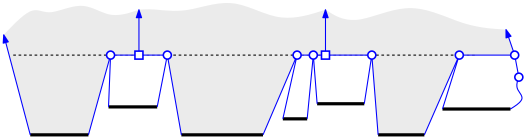
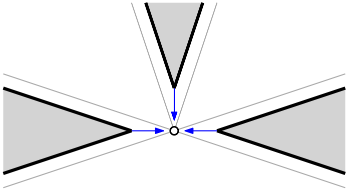


Figure : Resolution methods proposed in [Biedl et al., 2013].

Still open: How to handle split events?

Split events

The *standard scheme* works for unweighted straight skeletons.



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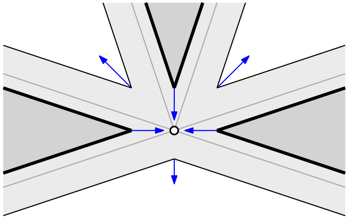
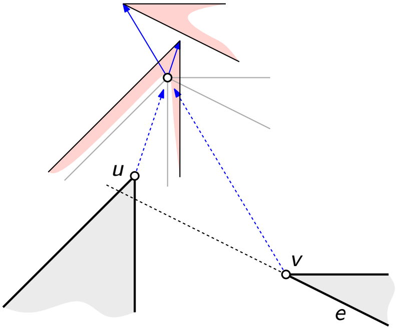
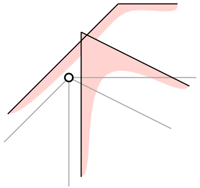


Split events

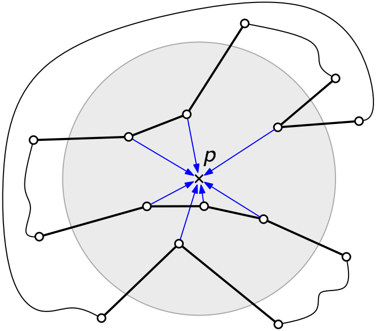
But for arbitrary weights the standard scheme may fail.





Split events

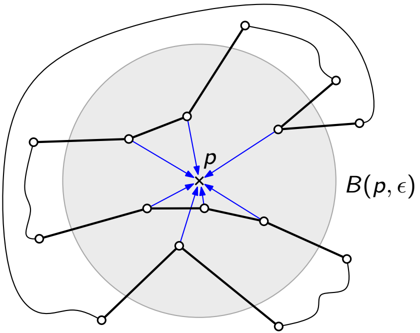
How to handle this?



Guiding principle

At all times between events, the wavefront shall be planar.

Pairing edges



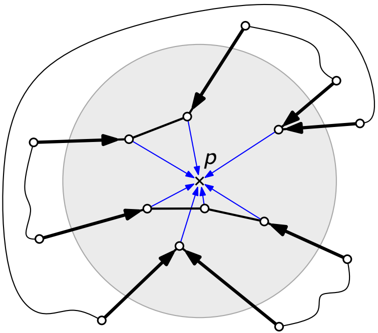
First:

- Remove collapsed edges.

Task: Find a pairing of remaining edges to restore planarity of $\mathcal{W}_P$.

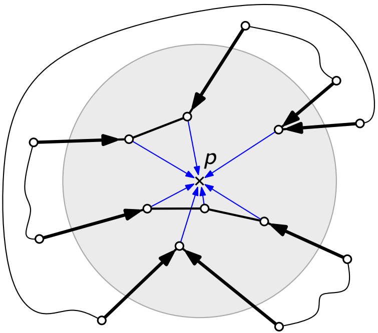
- Is this always possible? Uniquely?

Directed pseudo-line arrangements



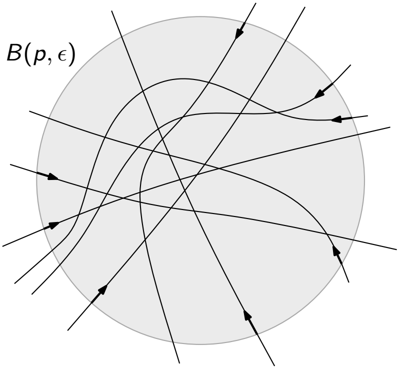
- We have k involved chains.
  - Hence, $2k$ (non-collapsed) edges.
  - Assign direction to each edge.

Directed pseudo-line arrangements



- We have k involved chains.
  - Hence, $2k$ (non-collapsed) edges.
  - Assign direction to each edge.
- Consider supporting lines of edges, after the event.
  - $\rightarrow$ pseudo-line arrangement $\mathcal{L}$ of directed pseudo-lines.

Planar matchings



- Every pair intersects, in a single unique point, within $B(p, \epsilon)$.

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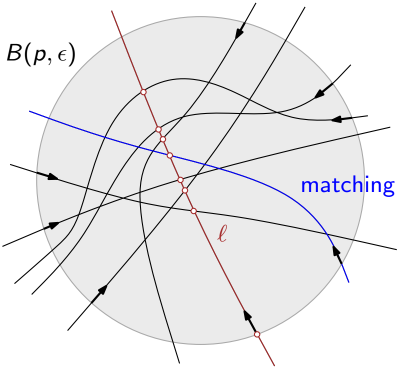

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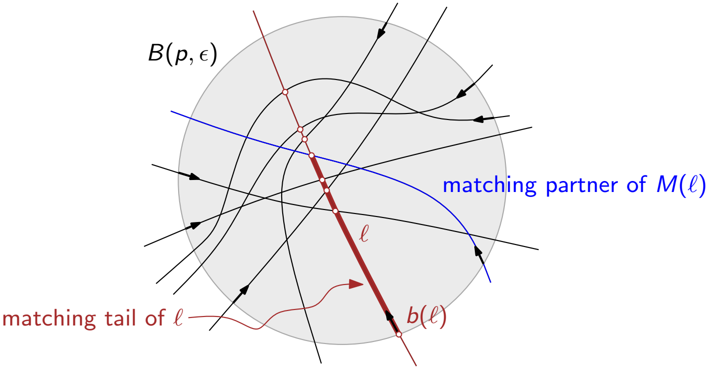
- Matching: grouping into pairs.

Planar matchings



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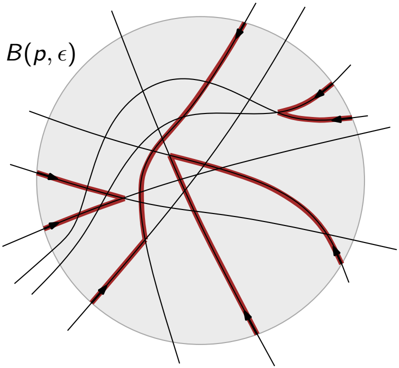
Planar matchings



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Planar matchings

Theorem

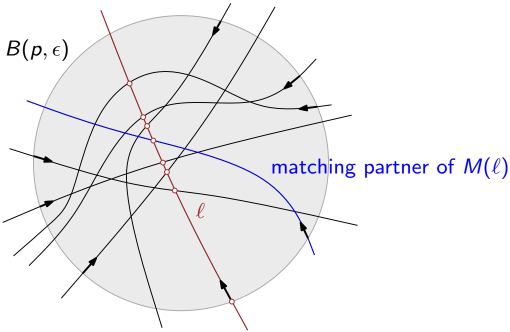
Every directed pseudo-line arrangement has a planar matching.

Stable roommates

- Every pseudo-line has a preference list (ranking) of all others.
- Blocking pair $\{\ell_i, \ell_j\}$: They prefer each other over their matching partners.
- Matching is stable if there are no blocking pairs.

Lemma

$\mathcal{L}$ has a planar matching if and only if there is a stable matching.



Stable partitions

Stable partition:

- Permutation π of $\ell_1, \ldots, \ell_N$.
- In each cycle of size ≥ 3: each ℓ prefers $\pi(\ell)$ over $\pi^{-1}(\ell)$.
- There is no party-blocking pair $\{\ell_i, \ell_j\}$: they prefer each other over $\pi^{-1}(\ell_i)$ and $\pi^{-1}(\ell_j)$.

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Theorem ([Tan and Hsueh, 1995])

1. *There is a stable partition, and it can be found in polynomial time.*
2. *There is a stable matching if and only if there is a stable partition with no cycles of odd size.*

Theorem

There are no parties of odd size for directed pseudo-line arrangements.

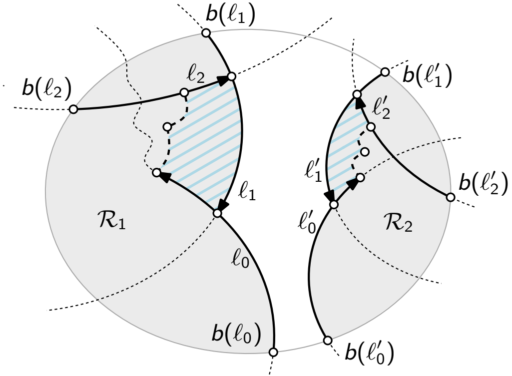
Odd parties do not exist

Lemma

The tails of ℓ and ℓ' do not intersect, unless $\pi(\ell) = \ell'$ or $\pi(\ell') = \ell$.

Lemma

There cannot be two parties of size at least three.



Acknowledgments

We would like to thank David Eppstein for mentioning this problem to us, and for the idea of interpreting the edge-pairing problem as a stable roommate problem.

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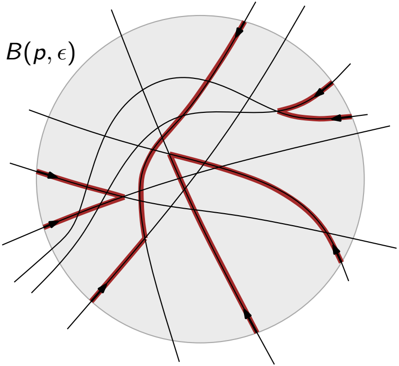
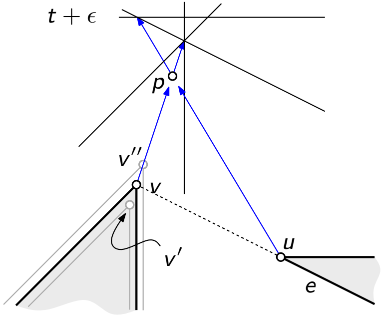
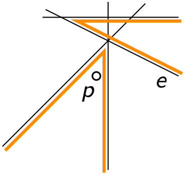
Non-Uniqueness



No planar wavefront



(a)



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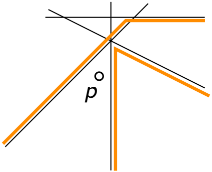

(c)