Weighted Skeletal Structures in Theory and Practice

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In geometry, a skeletal structure of a polygonal shape attempts to capture the essence of some geometric properties of the original shape. Tasks which are work-intensive or fragile to perform with just the polygonal shape can often be performed efficiently and robustly once an appropriate skeleton has been obtained. For instance, the medial axis of a polygon is such a skeleton, and constant-radius offset curves of a polygon are commonly computed by first constructing the polygon's medial axis.

This thesis presents the author's contribution to the field of computational geometry, in particular with respect to skeletal structures. We cover mitered offsets, which are related to the well-known constant-radius offsets, and variableradius offsets, where the distance to the input varies even along a single input segment. We study how the former can be obtained from straight skeletons, introduced by Aichholzer et al. two decades ago, and we introduce a suitable skeletal structure that encodes geometric information required to efficiently construct the latter.

Furthermore, this dissertation covers aspects of weighted straight skeletons. We discuss properties of multiplicatively-weighted straight skeletons in detail for different classes of input. We establish that, even in the presence of negative weights, they are always well-defined since events of the underlying wavefront propagation can be handled in all cases. We introduce additively-weighted straight skeletons, discuss their properties, and show how skeletons with both additive and multiplicative weights can be used in roof and terrain modeling.

Additionally, we present an algorithm to compute the positively-weighted straight skeleton for monotone polygons with n vertices in $O(n \log n)$ time.

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Part I

INTRODUCTION

PRELIMINARIES

1

This cumulative dissertation covers my research on weighted skeletal structures. The primary structure of interest in my work is the weighted straight skeleton. Additionally, we look into a second structure, a so-called variable-radius Voronoi diagram, in response to a specific application requirement where we find straight skeletons to be inapplicable.

First, I briefly introduce basic structures from computational geometry so that I can then summarize my work in Chapter 2. The main body of this dissertation is a selection of my work already published in or submitted to peer-reviewed conferences and journals. It can be found in Part II.

1.1 VORONOI DIAGRAM

The Voronoi diagram is a well-studied object in computational geometry. It is named after the Russian mathematician Georgy Feodosevich Voronoy and was introduced to computational geometry by Shamos [Sha75] and Shamos and Hoey [SH75] in the 1970's.

For a set S of points in the plane, called sites, the Voronoi diagram is the subdivision of the plane into regions, one for each site $s \in S$, such that all points in the region belonging to s are closer to s than to any other site of S [BCKO08]. Figure 1 shows a Voronoi diagram for a set of nine sites.

Voronoi diagrams have been generalized in different ways. Instead of the standard Euclidean metric, different metrics, such as the Manhattan metric (L_1) or the maximum metric (L_{∞}), can be used. Voronoi diagrams can not only be considered in the two-dimensional plane but also in 3-space or higher dimensional spaces. In fact, any metric space admits Voronoi diagrams. Another generalization is to extend the set of sites beyond just simple points to various types of geometric objects. Furthermore, different weights can be introduced to modify the metric on a per-site basis. Okabe et al. [OBSCoo] discuss these generalizations and several others in depth in Chapter 3 of their book.

1.1.1 Medial Axis

The medial axis was considered by Blum in the 1960's in the context of shape recognition [Blu67]. For a simple polygon P, it is the set of all points within P whose closest point on the boundary of P is not unique. Equivalently, it is the set of centers of circles within P that touch the boundary of P in two or more points [OBSCoo].



Figure 1: A Voronoi diagram of nine sites. The region belonging to the site s is highlighted.



Figure 2: A simple polygon (in black) with its Voronoi diagram (in blue). The medial axis of the polygon consists only of the solid blue segments.

The medial axis is a subset of the Voronoi diagram of the edge and vertex set of the boundary of P, and due to its structure it is sometimes also called the skeleton of P. Figure 2 shows a simple polygon and the Voronoi diagram of its edges and vertices in its interior relative to the standard Euclidean metric. The curves separating regions are line segments and parabolic arcs. Segments incident to reflex vertices of the polygon (vertices where the interior angle is larger than π) are not part of the medial axis.

1.1.2 Applications

The Voronoi diagram is a versatile tool, and *Nearest Neighbor* queries and *Closest Pair* are two well-known problems which can be solved efficiently once a Voronoi diagram is constructed. I refer to the survey by Aurenhammer [Aur91] and the book by Okabe et al. [OBSCoo] for more examples.

An application related to the work in this thesis is tool-path generation. In his book, Held [Hel91] describes in detail how to robustly compute Voronoi diagrams of polygonal structures. Based on the Voronoi diagram, he then constructs offsets which can be used as the path of a cutting tool, with no need for finding self-intersections and avoiding loop removals.



Figure 3: The Voronoi diagram allows for robust and efficient computation of offsets.

The approach bases on two key properties: Offset elements of a site s are contained in the Voronoi region of site s, and the combinatorial structure of the offset can be obtained directly from the structure of the Voronoi diagram.

Figure 3 shows a polygon, its Voronoi diagram, and several offset curves, one of them in bold and green. Offset segments in the Voronoi region of a vertex are circular arc segments, and offset segments in the region of a polygonal edge are line segments. Offset elements of different input sites meet on the bisector of these sites.

1.2 STRAIGHT SKELETON

About twenty years ago, Aichholzer et al. [AAAG95] introduced the straight skeleton to computational geometry. As the name suggests it also is a skeletal structure. However, unlike the medial axis, it comprises straight line segments only. The straight skeleton of a simple polygon P is defined as the result of a shrinking process of P. Briefly, the edges of P move inwards at unit speed. The shrinking polygon, called the wavefront, undergoes changes during this process to maintain planarity. In particular, in edge events, an edge is dropped from the wavefront when it collapses to zero length. In split events, the wavefront is split in two when a reflex vertex moves into an opposite wavefront edge. When all wavefront components have collapsed, the process ends. The straight skeleton then is defined as the geometric graph whose edges comprise the traces of all wavefront vertices during this shrinking process.



Figure 4: A simple polygon (in black) is undergoing a shrinking process. The vertices of the wavefront (gray and dashed) trace out the edges of the straight skeleton.



Figure 5: The roof induced by the straight skeleton from Figure 4.

Figure 4 depicts the same simple polygon as before in Figures 2 and 3. The dashed and gray polygons are the wavefront at different stages in the process.

Aichholzer et al. [AAAG95] also note that the straight skeleton of a polygon P induces a unique roof of P: First, P is embedded in the xy-plane. Then, a roof above P is constructed by lifting each point p in the interior of P by a value that corresponds to the orthogonal distance between p and the input edge that traced out the straight skeleton face in which p lies. (For points that lie on an edge or vertex of the straight skeleton any incident face can be chosen.) Figure 5 shows the roof corresponding to the straight skeleton in Figure 4.

This roof is a handy tool for proving various properties about straight skeletons, but it is also useful in itself for modeling terrains or actual roofs of buildings.

1.2.1 Straight Skeletons of Planar Straight-Line Graphs

While originally straight skeletons were only defined for simple polygons as input, Aichholzer and Aurenhammer [AA96] later extended the definition to permit planar straight-line graphs, i.e., graphs where edges do not intersect except at common endpoints.

Since there is no well-defined interior, input edges now send wavefront segments to both sides. At input vertices of degree one, one additional edge orthogonal to the incident input edge is inserted into the wavefront. The offsets and straight skeleton shown in Figure 6 illustrate this procedure.



Figure 6: Straight skeleton of a planar straight-line graph with families of offset curves.

1.2.2 Weighted Straight Skeleton

As a further generalization of straight skeletons, the weighted straight skeleton was first mentioned by Aichholzer and Aurenhammer [AA98] and then by Eppstein and Erickson [EE99]. It is defined by the same shrinking process except that edges no longer need to move all at the same speed; see Figure 7.



Figure 7: Polygon with a weighted straight skeleton. The edges marked with * move inwards at a speed of 3, those marked with \diamond move at a speed of 1/3. Unmarked edges move at unit speed.

1.2.3 Applications

Straight skeletons have a surprisingly diverse set of applications. Demaine et al. [DDL98] use straight skeletons in mathematical origami to solve the Cutand-Fold problem: Find a flat folding of a piece of paper, such that a single straight-line cut on the folding suffices to obtain one piece of a desired shape.

Tomoeda and Sugihara [TS12] apply straight skeletons to create signs with an illusion of depth. Sugihara [Sug13] also applies weighted straight skeletons to computer-aided creation of pop-up cards. (Pop-up cards are folded sheets of paper that produce meaningful 3D-structures when opened.)

In graph drawing, Bagheri and Razzazi [BR04] produce drawings of trees inside simple polygons and use straight skeletons to find a good distribution of the vertices.

In geographic information systems, Haunert and Sester [HSo8] note that based on straight skeletons, topology-preserving area collapsing can be performed. Vanegas et al. [Van+12] apply straight skeletons for generating parcels in urban modeling. Laycock and Day [LDo3] use straight skeletons for generating large 3D-models of urban environments based on building footprints. Kelly and Wonka [KW11] use weighted straight skeletons to procedurally construct a variety of different architectural surfaces.

Tănase and Veltkamp [TVo₃] and Aurenhammer [Auro8] employ (weighted) straight skeletons for polygon decomposition. Barequet et al. [BEGVo8] use multiplicatively-weighted straight skeletons as a theoretical tool for computing (unweighted) straight skeletons in three-space.

1.2.4 Computation

When introducing the straight skeleton, Aichholzer et al. [AAAG95] note that simulating the wavefront propagation process might work well in practice. For a polygon with n vertices, this approach runs in $O(n^3)$ time and linear space. By using a priority queue to store all possible events instead of computing the next event after every change in the wavefront, runtime complexity can be brought down to $O(n^2 \log n)$ at the cost of quadratic memory. This is essentially the algorithm implemented by Cacciola [Caco4] for CGAL, the Computational Geometry Algorithms Library [CGAL].

Aichholzer and Aurenhammer [AA98] observe that if one maintains a triangulation of the wavefront polygons, then each edge and split event is witnessed by the collapse of a triangle of this kinetic triangulation. The converse does not hold, i.e., a collapse of a triangle does not necessarily indicate an edge or a split event. Instead, it can indicate a so-called flip event which requires a local reconfiguration of the triangulation to maintain a valid tessellation of the wavefront polygon. Simulating the wavefront propagation this way, only linear space is needed. No tight bounds on worst-case runtime are known. The trivial worst-case upper bound stems from the fact that there are a linear number of wavefront vertices over time, these combine to at most $O(n^3)$ different triangles, and each triangle may collapse at most twice. These collapse times are maintained in a priority queue, resulting in a runtime of $O(n^3 \log n)$. Huber [Hub12] constructs input for which the algorithm will require $\Theta(n^2 \log n)$ time, leading to a lower bound of $\Omega(n^2 \log n)$ time in the worst case. We implemented Aichholzer and Aurenhammer's triangulation-based algorithm and found that in practice it runs in $O(n \log n)$ time for a wide range of real-world input [PHH12, PH15].

The currently best known algorithm for unrestricted input is due to Eppstein and Erickson [EE99]. Using advanced closest-pair data structures, they compute the straight skeleton of a simple polygon in $O(n^{17/11+\epsilon})$ time and space for any fixed $\epsilon > 0$. While Eppstein and Erickson only cover simple polygons, their approach will also work for arbitrary planar straight-line graphs [Hub12].

Eppstein and Erickson [EE99] also introduce the motorcycle graph in an attempt to isolate the difficulty posed by finding split events. Cheng and Vigneron [CV02, CV07] are the first to describe an algorithm based on motorcycle graphs. Let P be polygon with r reflex vertices, and let P be restricted to be in general position, such that all interior nodes of the straight skeleton will be of degree exactly three. In particular, no two reflex vertices are allowed to meet during the wavefront propagation process. With these restrictions, Cheng and Vigneron are able to compute the motorcycle graph induced by P in $O(r\sqrt{r}\log r)$ time and $O(r\sqrt{r})$ space [Hub12]¹. From the motorcycle graph, they obtain the straight skeleton using a randomized divide and conquer approach in $O(n \log^2 n)$ expected time, resulting in a total of $O(n \log^2 n + r \sqrt{r \log r})$ expected time to compute the straight skeleton of a simple polygon in general position. Cheng and Vigneron [CVo7] also extend their algorithm to handle polygons with holes. For a polygon with h holes, the algorithm constructs the straight skeleton in $O(n\sqrt{h+1}\log^2 n + r\sqrt{r}\log r)$ expected time. Recently, together with Mencel [CMV14], they improve the reduction of straight skeleton to motorcycle graph to $O(n(\log n) \log r)$ time. Combined with a new result for motorcycle graphs by Vigneron and Yan [VY14], the straight skeleton of a polygon in general position can now be constructed by a deterministic algorithm in $O(n(\log n) \log r + r^{4/3 + \epsilon})$ time. For unrestricted polygons, the motorcycle graph construction by Eppstein and Erickson [EE99] is still relevant, and computing the straight skeleton from a polygon runs in $O(n(\log n)\log r + r^{17/11+\epsilon})$ time for any fixed $\epsilon > 0$.

Huber and Held present another algorithm to construct the straight skeleton from the motorcycle graph [HH10, Hub12]. Like in the triangulation based approach by Aichholzer and Aurenhammer [AA98], they simulate the wavefront propagation and maintain a graph of the wavefront. Their core idea is to combine this graph with those parts of the motorcycle graph that have not yet been swept over by the wavefront. The kinetic graph thus obtained always consists of convex faces only, and thus any change in the topology correlates to an edge of the graph collapsing to a length of zero. These events are maintained in a priority queue and processed in order. Their algorithm has a worst-case complexity of $O(n^2 \log n)$. However, extensive tests performed with their implementation suggest a runtime of $O(n \log n)$ for practical applications.

¹ Huber discusses the space complexity of the algorithm in his thesis as Cheng and Vigneron did not cover this topic in their work.

1.2.5 Recognition

Graph recognition is the problem of determining, for a given graph G, if it is a graph of a certain type. This is easy for some graph classes, for instance for trees (*Is it connected? Yes. Is it free of cycles? Yes. Therefore, it is a tree.*), but it can become more complex quickly. Ash and Bolker [AB85] studied the problem of deciding whether a geometric graph is a Voronoi diagram.

Aichholzer et al. [Aic+12] consider the graph recognition problem for straight skeletons for trees with fixed cyclic order of incidences at each vertex. They note that each tree whose interior vertices have degree at least three can be realized as the straight skeleton of a convex polygon. They also consider trees where, in addition to the cyclic order, all edge lengths are fixed. They solve the problem for star graphs and for caterpillar graphs. They leave open the problem for arbitrary trees.

Biedl et al. [BHH13] consider the following variant: Given a geometric graph G, is it the straight skeleton of an unknown planar straight-line graph H? They provide necessary and sufficient conditions for G, and they also provide an algorithm to construct a graph H from G such that G is the straight skeleton of H.

In related research [Aic+15], we consider trees with directed edges. Given such a directed tree T, we construct a polygon P, if it exists, such that during the wavefront propagation of P, each edge e of T is traced out by a wavefront vertex along the direction of e. We give necessary and sufficient conditions that T needs to meet for such a polygon to exist. We then continue to solve the more complex problem for a tree T where, additionally, we assign to each edge of T a requirement that it be traced out by a convex or reflex wavefront vertex.

CONTRIBUTION

2.1 COMPUTING OFFSET CURVES

2.1.1 Mitered Offset Curves

Held [Hel91] demonstrates that given the Voronoi diagram of a polygonal shape, constant-radius offset curves can be computed efficiently and robustly. In *Computing Mitered Offset Curves Based on Straight Skeletons* [PH15] (page 26), we apply a similar approach to constructing mitered offsets.

Building upon the triangulation-based algorithm by Aichholzer and Aurenhammer [AA98], we extend our earlier work [PHH12] to develop means for correctly handling input that is not necessarily in general position even on real-world hardware with limited-precision arithmetic operations.

We present results of extensive performance tests using our implementation, and we compare offset curves produced by our code to those produced by other methods used in practice; see Figure 8.



Figure 8: [PH15]: An input polygon (black) with one offset curve. The offset in (a) is generated by both CLIPPER [Joh14] and GEOS [San+13], two well-known polygon-clipping libraries; the offset in (b) is produced by our approach and has been generated by our code, SURFER.

2.1.2 Variable Radius Offsets

Consider an offset curve, either a constant-radius offset such as induced by a Voronoi diagram or a mitered offset induced by a straight skeletons. Then, in some sense, all offset segments will be at the same, constant distance to the input. For constant-radius offsets that distance is the standard Euclidean distance, and for mitered offsets it is the orthogonal distance to the corresponding input segment. In *Generalized Offsetting of Planar Structures Using Skeletons* [HHP16] (page 28), we seek offset variants where the offset distance may vary, not only between the different input sites, but even along one input site. This is a requirement that cannot be met with standard or weighted Voronoi diagrams or weighted straight skeletons. Yet, we still strive for an underlying skeletal structure which enables robust and efficient generation of multiple families of such variable offsets.

We define and study what we call a *variable-radius Voronoi diagram*, which is a closest-site Voronoi diagram where the distance of a point in the plane to a line-segment site s is weighted differently for different points along s. Based on this diagram we can create the kind of offsets we seek; see Figure 9.



Figure 9: [PH15]: Variable Voronoi diagrams of input shapes with one offset curve. The size of vertex markers is proportional to their weight. Note how the offset distance varies along input segments that have non-constant weights.

2.2 WEIGHTED STRAIGHT SKELETONS

Several properties have been shown by Aichholzer et al. [AAAG95] for the unweighted straight skeleton of a simple polygon. For instance, the straight skeleton of a polygon is a tree, it partitions the interior of the polygon into faces monotone to their corresponding input edge, and it induces a roof that is a terrain and has no minima in its interior.

2.2.1 Multiplicatively Weighted Straight Skeletons

Even though, as summarized in Section 1.2.3, multiplicatively-weighted straight skeletons have been used for years in a large number of applications, very little investigation in the principal properties of weighted straight skeletons has been done. With *Weighted Straight Skeletons in the Plane* [Bie+15b] (page 30), we try to close this gap between theoretical understanding and practical needs. We study in detail which properties of the unweighted straight skeleton carry over to its weighted pendant for polygons and for polygons with holes; see Table 1.

Polygon with holes	$\sigma \equiv 1$	σ positive	σ arbitrary
S(P) is connected	√ Lemma 15	√ Lemma 15	× Lemma 12
S(P) has no crossing	✓ Lemma 6	√ Lemma 6	× Lemma 3
f(e) is monotone w.r.t. e	✓ as in [AAAG95]	× Lemma 11	× Lemma 11
bdf(e) is a simple polygon	✓ as in [AAAG95]	imes Lemma 8	imes Lemma 8
roof is z-monotone	✓ Lemma 4	√ Lemma 4	× Lemma 5
S(P) has $n + v - 1 + h$ arcs	✓ Corollary 17	✓ Corollary 17	× Lemma 3
S(P) is a tree	× Corollary 17	× Corollary 17	× Lemma 12

Table 1: [Bie+15b]: Results for a polygon with h holes, with n denoting the number of vertices of the polygon P and v denoting the number of nodes of the straight skeleton S(P).

We provide a non-procedural characterization of the roof and the straight skeleton of convex polygons with arbitrary weights, and we show that it can be computed in linear time.

As Kelly and Wonka [KW11] and Huber [Hub12] observed before us, the weighted straight skeleton is ambiguously defined when parallel wavefront edges of different weights become adjacent. We note a second ambiguity in the presence of negative weights concerning the pairing of wavefront edges after split events. In fact, it is not even clear that there always is a pairing of edges such that the wavefront is planar again after a split event.

2.2.2 Existence of Weighted Straight Skeletons

It is this last issue that we focus our attention on in *Planar Matchings for Weighted Straight Skeletons* [BHP14] (page 32). We look at directed pseudo-line arrangements, and introduce planar matchings on pseudo-lines. We then translate the problem of finding a planar matching into a stable roommates problem, and by using results by Tan [Tan91] and Tan and Hsueh [TH95] we are able to show that for our special case the stable roommates problem always has a solution. This solution translates back directly to yield all possible pairings of wavefront edges after a split event.

Any potential ambiguity of the wavefront after an event follows from the fact that from the solution of some stable roommates problems more than one planar matching can be obtained. Furthermore, using our framework we can now show that the weighted straight skeleton is in fact well defined and exists for all polygons and weights.

2.2.3 Additively Weighted Straight Skeletons

In the wavefront propagation process, multiplicative weights translate to wavefront edges moving faster or slower, or even towards the outside of the polygon if weights are negative. In *Straight Skeletons with Additive and Multiplicative Weights and Their Application to the Algorithmic Generation of Roofs and Terrains* [HP16] (page 34), we extend this concept and add additive weights, which translate to the movement of edges being delayed.

Like in previous work, we study the properties of the resulting additivelyweighted skeleton and show that it is well defined. We argue that multiplicative weights need not stay constant over time, but that any piecewise constant weight function for edge speeds will result in valid roofs and skeletons.

These concepts enable more versatile roof modeling as well as terrain generation; see Figures 10 and 11.



Figure 10: [HP16]: An additively-weighted straight skeleton and the roof of a house which it induces.



Figure 11: [HP16]: (a) A gablet roof, induced by the straight skeleton of a rectangle with appropriate weight functions. (b) The terrain induced by the straight skeleton of a planar straight-line graph of a river system. Faces of the terrain are at different slopes due to different multiplicative weights, and some faces start at a larger height due to additive weights.

2.2.4 Straight Skeletons of Monotone Polygons

The unweighted straight skeleton of a convex polygon coincides with the medial axis or Voronoi diagram, which Aggarwal et al. [AGSS89] show how to compute in linear time. In our work [Bie+15b] (see Section 2.2.1), we are able to extend this result to computing the weighted straight skeleton of convex polygons in linear time as well.

On the other hand, the approach of Eppstein and Erickson [EE99], which runs in $O(n^{17/11+\epsilon})$ time and space, is still the best known result for unrestricted input. For polygons with holes, the known lower bound is $\Omega(n \log n)$, which can be shown by reducing sorting to computing straight skeletons [Hub12]. No tight bounds are known for polygons, polygons with holes, or planar straight-line graphs.

In A Simple Algorithm for Computing Positively Weighted Straight Skeletons of Monotone Polygons [Bie+15a] (page 36), we present an algorithm to compute the straight skeleton of a monotone polygon with n vertices in $O(n \log n)$ time and linear space. We split the polygon into two monotone chains, and we observe that the wavefront propagation of each chain independently can be simulated in $O(n \log n)$ time as every change of the wavefront is witnessed by an edge-collapse. The two straight skeletons thus obtained can be merged into the straight skeleton of the polygon again in $O(n \log n)$ time; see Figure 12.



Figure 12: [Bie+15a]: A monotone polygon and its straight skeleton in blue. The straight skeleton of the lower chain is shown in red and dashed, the merge line between the straight skeleton of the upper and lower chain in black and dashed.

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Part II

PUBLICATIONS

Note:

The abridged version of this work does not contain full copies of the papers that comprise this cumulative thesis. Instead, only references are provided.

COMPUTING MITERED OFFSET CURVES BASED ON STRAIGHT SKELETONS

Peter Palfrader and Martin Held [PH15].

[[]PH15]: Peter Palfrader and Martin Held: Computing Mitered Offset Curves Based on Straight Skeletons. In Computer-Aided Design and Applications, volume 12(4), February 2015, pages 414-424.

DOI: 10.1080/16864360.2014.997637

[CADA-mitered.pdf]

GENERALIZED OFFSETTING OF PLANAR STRUCTURES USING SKELETONS

Martin Held, Stefan Huber, and Peter Palfrader [HHP16].

[[]HHP16]: Martin Held, Stefan Huber, and Peter Palfrader: *Generalized Offsetting of Planar Structures Using Skeletons*. In *Computer-Aided Design and Applications*, volume 13 (5), 2016. DOI: 10.1080/16864360.2016.1150718

[CADA-generalized.pdf]

WEIGHTED STRAIGHT SKELETONS IN THE PLANE

Therese Biedl, Martin Held, Stefan Huber, Dominik Kaaser, and Peter Palfrader [Bie+15b].

ERRATA. Franz Aurenhammer points out that footnote 5 (When \mathcal{P} is a pyramid with a very thin U-shaped base then determining the three-dimensional initial wavefront of the top vertex necessarily involves negative weights) is not correct.

Indeed, positive weights in the 2D straight skeleton suffice to compute the offset near a vertex in a 3D polyhedron.

Negative weights are, however, needed in the related problem of obtaining a vertex of a polyhedron \mathcal{P} and its incident faces as the roof of a polygon resulting from the intersection of \mathcal{P} with a plane near the vertex.

[[]Bie+15b]: Therese Biedl, Martin Held, Stefan Huber, Dominik Kaaser, and Peter Palfrader: *Weighted Straight Skeletons in the Plane*. In *Computational Geometry: Theory and Applications*, volume 48 (2), February 2015, pages 120–133. DOI: 10.1016/j.comgeo.2014.08.006

[CGTA-weighted.pdf]

PLANAR MATCHINGS FOR WEIGHTED STRAIGHT SKELETONS

Therese Biedl, Stefan Huber, and Peter Palfrader [BHP14].

[[]BHP14]: Therese Biedl, Stefan Huber, and Peter Palfrader: *Planar Matchings for Weighted Straight Skeletons*. In *Proceedings of the 25th International Symposium on Algorithms and Computation (ISAAC 2014)*. Lecture Notes in Computer Science, volume 8889, Springer-Verlag, December 2014, pages 117–127. DOI: 10.1007/978-3-319-13075-0_10

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STRAIGHT SKELETONS WITH ADDITIVE AND MULTIPLICATIVE WEIGHTS AND THEIR APPLICATION TO THE ALGORITHMIC GENERATION OF ROOFS AND TERRAINS

Martin Held and Peter Palfrader [HP16].

[[]HP16]: Martin Held and Peter Palfrader: *Straight Skeletons with Additive and Multiplicative Weights and Their Application to the Algorithmic Generation of Roofs and Terrains*. Submitted to Computer-Aided Design. 2016. arXiv: 1604.03362

[CAD-additive-skeletons.pdf]

A SIMPLE ALGORITHM FOR COMPUTING POSITIVELY WEIGHTED STRAIGHT SKELETONS OF MONOTONE POLYGONS

Therese Biedl, Martin Held, Stefan Huber, Dominik Kaaser, and Peter Palfrader [Bie+15a].

[[]Bie+15a]: Therese Biedl, Martin Held, Stefan Huber, Dominik Kaaser, and Peter Palfrader: *A Simple Algorithm for Computing Positively Weighted Straight Skeletons of Monotone Polygons*. In *Information Processing Letters*, volume 115(2), February 2015, pages 243–247. DOI: 10.1016/j.ipl.2014.09.021

[IPL-monotone.pdf]