Hamiltonian Triangulations and Triangle Strips: an overview

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Outline

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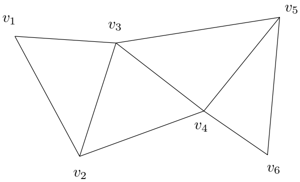
Motivation

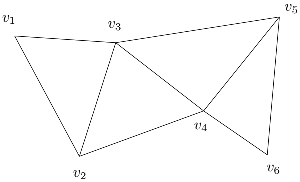
- We can transmit n triangles in $3n$ time/items.



- $(v_1, v_2, v_3), (v_2, v_4, v_3), (v_4, v_5, v_3), (v_4, v_6, v_5)$
- But can we do better?

Triangle Strips

- Yes, we can do better.
- Do not specify one triangle at a time, but **sequences** of triangles.



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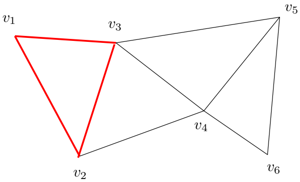

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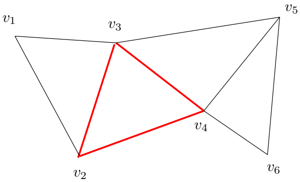

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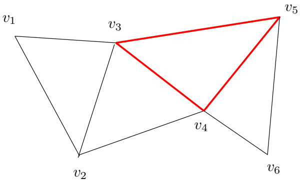

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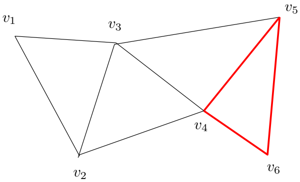

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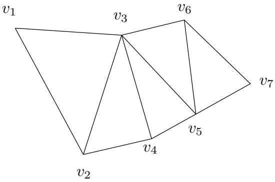
Non-sequential Strips

- Not all strips are sequential:



- $(v_1, v_2, v_3, v_4, v_5, \ldots$ and now?
- $(v_1, v_2, v_3, v_4, swap, v_5, v_6, v_7)$, or
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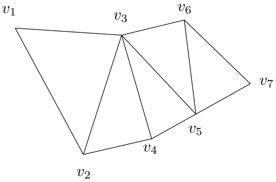

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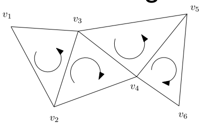
Visibility

- Sometimes triangle orientation is used for backface culling.



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- Solution: Use swap primitive or zero-area triangle.

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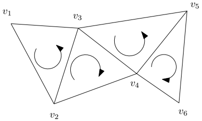

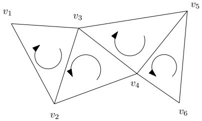

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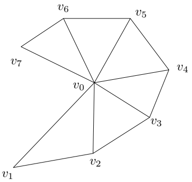
Triangle Fans

- Similar construct: Triangle Fan.



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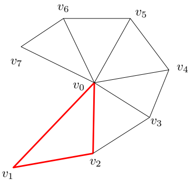

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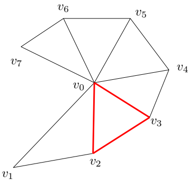

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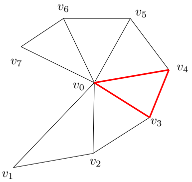

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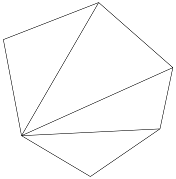
Triangle Fans

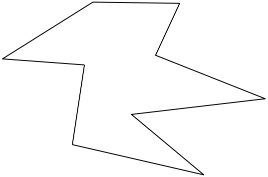
- Every convex polygon can be turned trivially into a triangle fan.



Outline

Triangulation of a Polygon

- Polygon triangulation is the decomposition of a polygonal area into a set of triangles [WP].



- Many different algorithms to create a triangulation:
  - Ear-clipping: $\mathcal{O}(n^2)$
  - Convex polygons: $\mathcal{O}(n)$
  - all simple polygons: $\mathcal{O}(n)$ [Chazelle 1991]
  - non simple polygons: $\Omega(n \log n)$.

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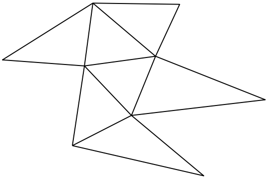

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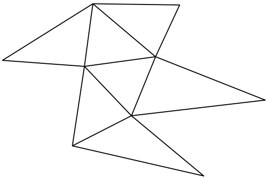
Dual Graph of a Triangulation

- A graph $\mathcal{G}$ is said to be the **dual graph** of a triangulation $\mathcal{T}$ if each vertex of $\mathcal{G}$ corresponds to exactly one triangle of $\mathcal{T}$, and two vertices are connected iff their corresponding triangles share an edge.



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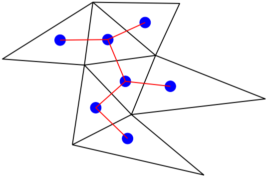
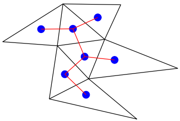
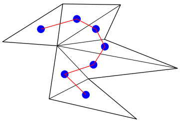


Hamiltonian Triangulation

- Def.: A triangulation is **Hamiltonian** if its dual graph contains a Hamiltonian path [Arkin et al., 1994].



not a Hamiltonian triangulation



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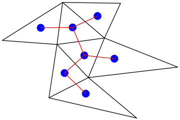

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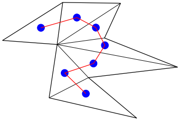

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Hamiltonian Triangulation

- Problem: Does a given simple polygon P have a Hamiltonian triangulation?
- Arkin et al. solve this in $\mathcal{O}(|visibilitygraph|)$:
  - Def.: $D[i,j] \Leftrightarrow$ subpolygon left of (i,j) has a Hamiltonian triangulation ending with (i,j).
  - $D[i,i+2] = (i,i+2)$ is a chord of P.
  - $D[i,j] = (D[i,j-1]$ and $(i,j-1)$ visible$)$ or $(D[i+1,j]$ and $(i+1,j)$ visible$)$.
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[Arkin et al., 1994]

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[Arkin et al., 1994]

Two-Guard-Problem

- Walkability:
  - P is **walkable** if two guards can walk along different paths from a point s on the polygon to a point t on the polygon without losing sight of each other.
  - P is **straight walkable** if the guards need not backtrack.
  - P is **discretely straight walkable** if at any time only one guard is walking while the other rests at a vertex.

Two-Guard-Problem

- Easy to see: a discretely straight walkable polygon has a Hamiltonian triangulation [Arkin et al., 1994].
- Walkability, Straight Walkability and Discretely Straight Walkability can be checked in $\mathcal{O}(n)$ time [Bhattacharya et al., 2001]. Furthermore, Bhattacharya et al.'s algorithm actually determines all pairs of points of P which allow ((discrete) straight) walks.
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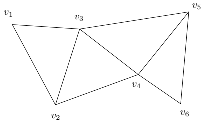
Polygons that are not simple

- Not so easy for polygons with holes.
- The decision problem of whether a given polygon has a Hamiltonian triangulation is proven to be $\mathcal{NP}$-complete (reduced from problem of whether a planar, cubic graph is Hamiltonian) [Arkin et al., 1994].

Sequential Triangulations

- A triangulation is said to be sequential if its turns all alternate left and right.



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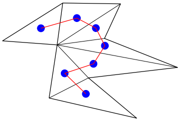

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- A triangulation is sequential if its dual graph contains a Hamiltonian path such that no three triangulation edges consecutively crossed by the path share a triangulation vertex [Arkin et al., 1994].
- No *swap* primitive or zero-area triangles needed to encode the triangulation.

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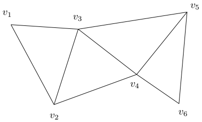

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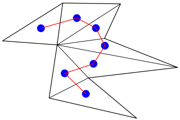

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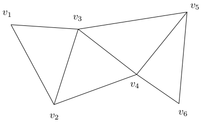

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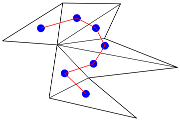

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Sequential Triangulations

- Testing if a given triangulation is sequential can be done in linear time [Arkin et al., 1994].
- Generating a sequential triangulation of a simple polygon can be done in $\mathcal{O}(n \log n)$ [Flatland 2004].

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Tristrip Decomposition

- Not every triangulation can be given as a single strip.
- But can we reduce a triangulation to a number k of strips.
- Only $n + 2 \cdot k$ vertices need to be transmitted then.
- Ideally we would use the smallest possible k.
- Unfortunately finding this k and its decomposition is $\mathcal{NP}$-hard [Estkowski et al., 2002].
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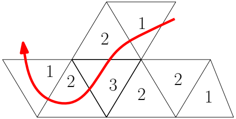
Improved SGI Stripping Algorithm

1. Pick a starting triangle.
2. Build three different strips, one for each edge of the triangle.
3. Prefer triangles that are adjacent to the least number of neighbors, look ahead in case of ties.
4. Extend these triangle strips in the opposite direction.
5. Choose the longest of these three, discard the others.
6. Repeat until all triangles are covered.

Can be implemented in linear time.

[Real-Time Rendering, 2nd ed., page 460], [Evans et al., 1996]



Fast and Effective Stripification of Polygonal Surface Models

- Three phases:
  1. Compute a spanning tree of the dual graph.
  2. Partition the tree into tristrips (path peeling).
  3. Concatenation phase: join small strips.

[Xian et al., 1999]

Efficient Generation of Triangle Strips from Triangulated Meshes

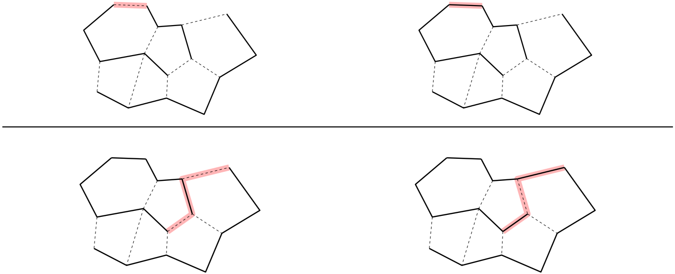
Def.: a *free* triangle is a triangle that does not belong to any strip; the *degree* of a triangle is the number of free neighbors.

- Start with a triangle with the lowest degree.
- For picking the next triangle, prefer higher degrees.
- Two strategies for tie breaking:
  1. Look ahead for triangles with degree 0, or degree 1 with a free neighbor of degree 1.
  2. Favour neighbors that do not require insert of a *swap*.
- Efficient data structures.
- Faster than FTSG, fewer strips but more vertices (in the OpenGL model).

[Kaick et al., 2004]

Iterative Stripification Algorithm

- Works on the dual graph of the triangulation.
- Uses a so-called *tunnelling* operator.



- Allows repair of triangulations that are damaged due to changes in topology (e.g. refinement).

[Porcu, Scateni 2003]

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References I

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Thank you for your attention.

Questions?